1 Appendix: Measuring jump height

The purpose of this appendix is to explain how jump height is derived from recorded vertical acceleration.

In order to calculate jump height from vertical acceleration as a function of time we need to revisit the equations of motion, which we will derive in the next section for sake of completeness.

2 Motion with known acceleration

If we know the acceleration as a function of time, initial position, and initial velocity we can calculate velocity and position as a function of time as well. This is because we can integrate acceleration with respect to time to get the change in velocity. This is necessary as acceleration is by definition the rate of change of velocity, and therefore summing up the rates of change of velocity with respect to time will be equivalent to the change in velocity. Hence, if we know the initial velocity we will then know the velocity at any given point in time by adding the integral of acceleration up to that point in time. Thereafter we can repeat the same procedure to get from velocity to position.

We will start with velocity based on acceleration. That is,

$$v(t) = v_0 + \int_{t_0}^{t} a(t)dt$$

where initial velocity (v_0) and acceleration as a function of time ($a(t)$) are known. For simplicity we will consider the flight phase where acceleration is constant, that is $a(t) = g = -9.81m/s^2$. The integral of a constant is the constant multiplied by time and an integration constant, that is $v(t) = gt + C_1$. It is relatively straightforward to presume that the integration constant must be the initial velocity, that is $C_1 = v_0$, and combining the two we get $v(t) = v_0 + gt$.

We will next use similar reasoning to obtain position from velocity as a function of time. That is, velocity is the rate of change of position and therefore the integral of velocity with respect to time is the change in position. Again, position is going to be the initial position + the integral of velocity up to that point in time and therefore,

$$x(t) = x_0 + \int_{t_0}^{t} v(t)dt$$

The integral of velocity with respect to time is $\int_{t_0}^{t}(\int_{t_0}^{t} a(t)dt + v_0 t)dt + C_2$, and we can set C_2 to x_0 like we did for velocity. We will keep on considering movement with constant acceleration and therefore (following integration rules) we get :

$$x(t) = x_0 + v_0 t + \frac{1}{2}gt^2$$

With this we have derived the equations of motion with constant acceleration. Deriving the formulae of motion with constant acceleration is actually not fully defined due to the integration constants and we had to use reason to give values to the integration constant. The fully determined way of deriving these equations is by starting from position as a function of time and taking the derivative with respect to time. That is,

$$v(t) = x'(t)$$

$$a(t) = v'(t) = x''(t)$$

We used the opposite in the above formula because we aim to determine jump height (position) based on measured acceleration. We can generalize the above approaches to our sampled data by realizing that calculating the definite integral of a discrete function (= a time series of sampled accelerations) is simply the cumulative sum of the values multiplied by the sampling interval. Starting from $v_0 = 0$ and $x_0 = 0$ (i.e. standing still, and setting the current height as the initial position) we will get

$$v(t) = \sum_{t=t_0}^{t} a_t dt$$

$$x(t) = \sum_{t=t_0}^{t} v_t dt$$

With these last two equations we are now able to derive velocity and position as a function of time from our sampled accelerations. This is easy to implement with any analysis programme, particularly so with Octave (or matlab) as they come with a function cumtrapz, which is meant for this particular application. Instead of giving the cumulative sum, the function uses trapezoidal integration, which is considered a more appropriate way to evaluate discrete integral compared to direct cumulative sum. That is, you would use

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velocity = cumtrapz(acceleration)*dt;
position = cumtrapz(velocity)*dt;
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in Octave or matlab to obtain velocity and position from your sampled acceleration. N.B. in practice, this would most likely lead into very poor results due to integration drift but that is besides the scope of this text.

3 Obtaining jump height from flight time

Now that we have defined the equations of motion we can return to the question of how do we get jump height from recorded accelerations. Firstly, we need to realise that during flight we are considering motion with constant acceleration $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$. We will quickly realise that it takes half of the flight time to get to the apex of the jump and therefore we are interested in $x(t_1/2)$, where $t_1 = flighttime$. This means that we have two unknowns ($x(t_1/2)$ and v_0), and only one equation. We can obtain a second equation by realising that $x(t) = 0$ for both $t = 0$ (take-off), and $t = t_1$ (touchdown), which gives us three equations and two unknows. This means we have an overdetermined group of equations, which is good news as an overdetermined group of equations can be solved.

$$x(t_0) = x_0 + v_0 t_0 + \frac{1}{2} a {t_0}^2; x(t_0) = 0, t_0 = 0, x_0 = 0, a = g, v_0 = ?$$

$$x(t_1/2) = x_0 + v_0 t_1/2 + \frac{1}{2} a t_1/2^2; t_1 = flighttime, x(t_1/2) = ?, a = g, v_0 = ?$$

$$x(t_1) = x_0 + v_0 t_1 + \frac{1}{2} a {t_1}^2; t_1 = flighttime, x(t_1) = 0, a = g, v_0 = ?$$

We can just disregard the first equation (because any v_0 would do) and use the remaining two equations (to have a group of equations with a unique solution). We will then first solve the third equation for v_0.

$$x(t_1) = x_0 + v_0 t_1 + \frac{1}{2} a {t_1}^2; x(t_1) = 0, x_0 = 0, a = g, v_0 = ?$$

therefore, $v_0 = \frac{1}{2} a {t_1}^2 / (-t_1) = -g t_1/2$. We will then insert the $v_0 = -g t_1/2$ into the second equation.

$$x(t_1/2) = x_0 + (-g t_1/2) \times t_1/2 + \frac{1}{2} a t_1/2^2; t_1 = flighttime, x_0 = 0 x(t_1/2) = ?, a = g$$

We will substitute the known values in to get

$$x(t_1/2) = (-g t_1/2) \times t_1/2 + \frac{1}{2} g t_1/2^2$$

and then simplify

$$x(t_1/2) = -g t_1/2^2 + \frac{1}{2} g t_1/2^2 = -\frac{1}{2} g t_1/2^2 = -\frac{g t^2}{8}$$

Note that we have implicitly defined g = -9.81 but using g = +9.81 we can take out the - sign from above and this is how we end up with jump height = $\frac{gt^2}{8}$ you will find reported in textbooks, literature, and the associated paper.

4 Obtaining jump height from take-off velocity

We have defined the equations of motion, and instead of taking the ight time based on the acceleration recording, we could start from standing still, and integrate vertical acceleration until the take-off instant to estimate take-off velocity using the two discrete sums we defined earlier. Since we have movement with constant acceleration during flight (neglecting air drag as negligible due to the relatively low velocity), we can derive the equation for jump height based on take-off velocity. That is, we will again obtain a group of equations based on the known facts. We know the take-off velocity, and we know that at the apex velocity is zero. Therefore, we can use the equation of velocity with con- stant acceleration for the first equation, and position with constant acceleration as our second equation evaluating both at the instant of the apex of the jump. That is;

$$v(t_a) = v_0 + at; v(t_a) = 0, a = g, v_0 = v_0, t_a = ?$$

$$x(t_a) = x_0 + v_0 t_a + \frac{1}{2} a {t_a}^2; x_0 = 0, t_a = time\,at\,apex, x(t_a) = ?, a = g, v_0 = v_0$$

We will solve the first equation for t_a by reorganising $t_a = -v_0/g$ and substitute the solution of t_a into the second equation;

$$x(t_a) = v_0 \frac{-v_0}{g} + \frac{1}{2} g (\frac{-v_0}{g})^2 = \frac{-v_0^2}{g} + \frac{1}{2} g \frac{v_0^2}{g^2} = \frac{-v_0^2}{g} + \frac{v_0^2}{2g} = \frac{-v_0^2}{2g}$$

Again, we note that we have implicitly defined $g = -9.81 m/s^2$, which cancels out the minus sign and we are left with $jump\,height = \frac{v_0^2}{2g}$ with $g = 9.81$, which you will find from the literature and the associated paper.

5 Obtaining jump height from concentric net impulse

Obtaining jump height based on concentric net impulse is akin to obtaining jump height based on take-off velocity. In order to get there, we need to know that the change in momentum is equivalent to the applied impulse. Momentum is $M = mv; M = momentum, m = mass, v = velocity$, and impulse is $I = ft; I = impulse, f = force, t = duration\,of\,force\,application$.

In order to get impulse from vertical acceleration we need to remember Newton's second law, which states $F = ma$. Now we will substitute F into I; $I = mat$, and integrate with respect to time from start of the concentric phase until take-off. We will then set $\Delta M = I$, i.e. $mv_{to} - mv_0 = mat; v_{to} = take$-off $velocity, v_0 = 0$. We will note that each term has m so we can just divide the m away to get $v_{to} = at$, and for variable acceleration we can replace at with $\sum_{t=t_0}^{t_{to}} a_t dt$. We can then just insert the take-off velocity into the equation we derived for obtaining jump height from flight time. If we actually have measured impulse (e.g. with a force plate) we will simply divide the impulse with body mass to obtain take-off velocity, or in other words $jump\,height = \frac{I^2}{2gm^2}$.